

Movements in the Term Structure of Interest Rates

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BOND PRICES TEND TO MOVE TOGETHER. STOCKS TEND TO GO THEIR OWN WAY. THIS DISTINCTION HAS IMPORTANT IMPLICATIONS FOR MANAGING THE RISKS ASSOCIATED WITH HOLDING THESE SECURITIES.¹

Because so much of the movement in stock prices is idiosyncratic, or security-specific, it is impossible to use one stock, or even a portfolio of stocks, to hedge the price movements in any other stock. For this reason, diversification, which helps minimize idiosyncratic risk, plays an important role in modern equity portfolio management. Only the relatively small proportion of movement in stock prices that is systematic, or common, across all stocks can be hedged using contracts whose payoffs are tied to the value of a stock market index, such as the S&P 500 futures contract traded on the Chicago Mercantile Exchange.²

In contrast, because so much of the movement in bond prices is systematic, bond portfolio management focuses on techniques for eliminating the common factor of interest rate risk by balancing short and long exposures to fluctuations in interest rates. Doing so does not require that portfolios be well diversified.³ It is possible to use a few bonds of differing maturities to hedge the price fluctuations in any single bond or portfolio of bonds. Alternatively, interest rate derivatives, such as the ten-year Treasury note futures contract traded on the Chicago Board of Trade, whose payoff is tied to the value of the ten-year Treasury note at the

expiration of the contract, may be used to hedge interest rate risk. Using derivatives rather than other bonds for hedging is generally more cost efficient. The futures markets are usually more liquid than the underlying bond markets, and short positions may be easily taken without the complications and costs of shorting securities. The enormous volume of trading in interest rate futures attests to the widespread use of such techniques.

Hedging to reduce or eliminate the common factors influencing the value of an interest rate-sensitive portfolio requires a model of how interest rates behave. This model may be a formal equilibrium- or arbitrage-based model, or it may be an ad hoc statistical model.⁴ The most widely used method for hedging bond portfolios is duration immunization, which matches the Macaulay duration of assets and liabilities. Macaulay duration is predicated on the assumption that interest rates for all maturities move up and down in parallel. Clearly, they do not do so. Nonetheless, Macaulay duration hedging is still widely used. Numerous studies have developed more advanced hedging models aimed at capturing changes in the shape of the term structure as well as changes in the overall level of interest rates.

This article first reviews these earlier studies, including those showing that term structure movements can be decomposed into three components, called factors. The empirical analysis then shows that the nature of this decomposition has been consistent since 1970 and that the structure of the factors has not changed appreciably even though interest rate volatility has. The investigation then turns to the time series behavior of the factors. Following this analysis, the implications for hedging and interest rate modeling are discussed. A numerical example demonstrates that hedging based on factor decomposition is superior to hedging based on traditional methods.

A Review of the Literature

Numerous articles compare various duration measures, including Macaulay duration. For example, a collection by Kaufman, Bierwag, and Toevs (1983) presents a number of studies comparing various duration specifications. These, in general, find that Macaulay duration performs as well as other linear models relating price and yield changes, although simple maturity did nearly as well in some cases. Ingersoll (1983) developed a measure based on the single-factor Cox-Ingersoll-Ross model and found it promising. However, Gultekin and Rogalski, using actual bond data rather than fitted yield changes, concluded that “the data are not consistent with the hypothesis that price and volatility of Treasury securities is adequately measured by simple duration” (1984, 252–53). They found that a multifactor duration hedge, based on a factor decomposition such as this article presents, outperformed the single-factor duration measures previously proposed whether based on a theoretical model or on

ad hoc assumptions. Ilmanen’s (1992) finding that the performance of duration as a measure of interest rate risk varied through time helps to explain the differences in previous studies.

Formal interest rate models are based on assumptions about the number of sources of uncertainty and their structure. It is possible to turn the process around and first ask how many factors underlie movements in the term structure, without specifying the exact nature of the relation between the factors and movements in bond prices beforehand. Litterman and Scheinkman (1991) used factor analysis (discussed below) to determine the number of the factors underlying movements in interest rates and their economic interpretation. They determined that three factors explain the majority of movements in interest rates for various maturities. Knez, Litterman, and Scheinkman (1994) used the same technique to examine short-term (less than one year) interest rates across a variety of money market instruments and found, surprisingly, that four factors are important. This anomalous result is explained in part by the mix of security types—Treasury bills, repurchase agreements, commercial paper, and bankers’ acceptances—which may not all have the same risk.⁵

Hedging to reduce or eliminate the common factors influencing the value of an interest rate sensitive portfolio requires a model of how interest rates behave.

1. For instance, the draft for the recently implemented Basle Accord on the use of internal models for risk-based capital assessment proposed requiring that “[e]ach yield curve in a major currency must be modeled using at least six risk factors. . . .” (Joint Notice 1995, 38086; emphasis in the original). The evidence in this and numerous other studies is that there are only three or four factors in some markets. Requiring models to incorporate six or more factors may well create the interesting problem of identifying the extraneous factors to be included.
2. Dynamic hedging of individual stock price movements using stock-specific options, if they exist, is theoretically possible if volatility is constant. However, under most pricing theories stock prices respond to changes in the systematic component of stock-price risks while options prices respond to changes in total risk, which is the sum of systematic and idiosyncratic risk. Thus, changes in the volatility of the idiosyncratic component of a stock’s return will not affect stock price but will affect the value of options written on the stock. If volatility changes cannot be hedged, then the effectiveness of the stock/option hedge will be reduced.
3. Although diversification has little role in managing interest rate risk, it is still important in managing credit or default risk if the bond portfolio consists of risky debt.
4. Examples of equilibrium-based interest rate models include the Cox, Ingersoll, and Ross (1985), or CIR, model and the widely used Vasicek (1977) model—both usually implemented as single-factor models. The Longstaff and Schwartz (1992) model is an example of a multifactor equilibrium model. These models all begin by modeling the process for the instantaneous interest rate and then using equilibrium arguments to derive the implied structure and evolution of the entire term structure. Arbitrage-based models, such as the Heath, Jarrow, and Morton (1990, 1992) model, take the term structure as given and model its evolution. These are naturally multidimensional, though single-factor restricted versions can be constructed.
5. Duffee (1996) has pointed out that Treasury bills of one month or less to maturity appear to show price movements that are idiosyncratic, that is, unrelated to changes in other interest rates, either those of longer-maturity Treasury bills or similar-maturity money market rates.



The studies by Litterman and Scheinkman and by Knez, Litterman, and Scheinkman both examine changes in interest rates rather than the levels of interest rates or changes in bond prices. For hedging purposes, it is not the levels of interest rates that are important but the changes, which in turn produce changes in bond prices. Most bonds are coupon bonds and are therefore portfolios of many different individual cash flows, each responding to a different zero-coupon bond yield change. Once the movements in zero-coupon yields are understood, the movements in coupon bond prices may be easily expressed as a function of these.⁶

Of Litterman and Scheinkman's three factors, the first one accounted for an average of 89.5 percent of the observed variation in yield changes across maturities. This factor, which they identified as a "level change" factor, helps to explain why Macaulay duration is so successful. While changes in levels are not the whole story, they are such a large part of what goes on in

interest rate movements that the assumption underlying Macaulay duration (that is, parallel movements up and down in interest rates) is a good first approximation. Nonetheless, Litterman and Scheinkman show that hedging based on three factors will improve hedge performance relative to Macaulay duration-based hedging by 28 percent on average and in some cases much more.

Empirical Analysis

What Are "Factors?" Factor analysis assumes that changes, in this case in interest rates, are driven by a few sources of variation that affect all interest rates to varying degrees. These sources of variation, in turn, summarize changes in the economy. Economic variables that may or may not affect interest rates include (along with innumerable others) the supply and demand for loans, announcements of unemployment and inflation, and changes in market participants' risk aversion arising from perceived changes in the prospects for continued economic growth. The key assumption of factor analysis is that this multitude of influences, which change interest rates continually, can be compactly summarized by a few variables, called factors, that capture the changes in the underlying determinants of interest rates. That thousands of influences can be boiled down to a few inputs into a compact, or

parsimonious, model is a very strong assumption. Part of the process of performing a factor analysis is to examine just how reasonable that assumption is.

As this article demonstrates, the assumption is quite reasonable for interest rate changes, but it is less so for stock returns. The relation among the underlying changes in the economy, economic agents' reactions to these changes, and the factors extracted by a factor analysis of changes in interest rates are not necessarily explicit. Factor analysis is a purely statistical description of the data. The factors obtained by such an analysis summarize the changes in interest rates (or stock returns or other variables) compactly. Interpreting the extracted factors and relating them to possible causal economic events is a separate challenge.

Because it is impossible to predict these underlying economic fluctuations completely, and hence the factors that summarize them, from one period to the next, the factors may be thought of as "shocks" to the term structure.⁷ By construction, over the sample period used to estimate the factor decomposition, each factor has an expected value of zero each month and a standard deviation of one. In any given period, the factor can take on any positive or negative value. Also, by construction, the factors are not correlated with each other in each period. Each factor has an impact on changes in interest rates, but the degree of that impact may vary across the term structure. Factor analysis describes the way each factor affects (or "loads on") each interest rate. The relations between interest rate changes and the factors are called factor loadings.⁸

Data and Methodology. The raw data used in this study are the prices of bills, notes, and noncallable bonds found in the monthly Government Bond Files produced by the Center for Research in Securities Prices (CRSP) at the University of Chicago. The analysis began with computation of a zero-coupon, or discount rate, term structure from the prices of bills, notes, and bonds (excluding those with embedded options) using a term structure estimation method developed in Fama and Bliss (1987).⁹

Data used are for the period from January 1970 through December 1995. Prior to 1970 insufficient numbers of eligible long-maturity bonds were available for computing usable term structures. Over time, there is more variation in the shape of the term structure at shorter maturities than at longer maturities. For this reason, ten unevenly spaced maturities are used in this study, namely, three and six months and one, two, three, five, seven, ten, fifteen, and twenty years to maturity. The Fama-Bliss yields each month at these horizons were differenced to compute the month-to-month yield changes used in this study.

Historical Performance of Factor Models. Before analyzing the factors themselves, it is useful to examine their ability to explain interest rate movements through

Factor analysis assumes that changes, in this case in interest rates, are driven by a few sources of variation that affect all interest rates to varying degrees.

time and to compare this with the ability of a similar number of factors to explain stock movements. For comparison purposes the stock returns were gathered from the CRSP Monthly Stock Returns File. The stocks selected were required to have no missing data during the 1970–95 period. Of the stocks meeting this requirement, ten were selected to cover a broad range of market values.

Beginning with the 1970–71 period a three-factor model was fitted to the twenty-four months of yield-change data. The cumulative percentage of variation in the observed data for one, two, and then three factors for that period was computed and plotted. The twenty-four-month window was then advanced one month and the process repeated until the last window covered the period from 1994 through 1995. The same procedure was applied to the ten stock returns. The results appear in Chart 1.

There is a clear difference in the ability of a few factors to explain changes in interest rates versus stock returns. A single factor, as yet unidentified, explains at least 60 percent of the observed variation in interest rate changes since 1970 and at least 78 percent since 1978. For stocks, however, the maximum variation in returns explained by a single factor is only 58 percent. Including two additional factors raises the interest rate variation explained to a minimum of 86 percent overall and 95 percent since 1978. Adding two factors raises the stock return variation explained to a maximum of only 84 percent.

The ability of a few factors to explain changes in interest rates is remarkably constant. During the period from 1971 through 1978, explanatory power was somewhat

lower than in subsequent periods.¹⁰ Since 1978 the yield-change variation explained by three factors has remained between 95 and 98 percent. In the same period the stock return–related figures varied from 63 to 80 percent.

The relatively moderate ability of a few linear factors, even though not tied to any particular theory, to explain stock returns underscores the poor performance of specific stock return models. For example, the capital asset pricing model (CAPM) developed in Sharpe (1964) and Linter (1965) hypothesizes that there is a single factor underlying stock returns and identifies that factor as “the market,” usually measured using a broad index of stocks. The CAPM, or the closely related market model, is able to explain between 1 and 60 percent for individual stocks.¹¹ Another theory of stock returns, the arbitrage pricing theory (APT) developed by Ross (1976), does not specify ex ante the nature or number of factors underlying these returns and so is similar to factor analysis in its application. Studies that have applied the APT, for example, Dhrymes, Friend, and Gultekin (1984), have found an ambiguous number of factors underlying stock returns, with the number of factors tending to increase with the number of stocks being analyzed. Because of the poor ability of stock return models to explain individual stock performance, most studies of stock return models are performed using portfolios of stocks. Portfolios tend to reduce the idiosyncratic component in returns and thus increase the importance of the common factors.

6. Bond prices are not solely a function of the term structure of interest rates. Bliss (1997) shows that, regardless of the term structure estimation used, there is a discrepancy between actual bond prices and the prices fitted using a term structure. However, Bliss also shows that these errors tend to be persistent, as one would expect if they resulted from nonpresent value factors such as liquidity or tax effects. Because pricing errors persist, they have an even smaller effect on bond returns than on bond prices. To check for the impact of bond-specific, versus term structure-specific, components in bond returns, the actual one-month returns for all bonds used in this study were regressed against the returns predicted solely by changes in the term structure. The resulting regressions show that changes in the term structure of zero-coupon yields explained 99.9 percent of the variation in actual returns. Thus, hedging based on changes in the term structure of zero-coupon yields is, for all practical purposes, sufficient to hedge actual bills and coupon-bearing notes and bonds.
7. This usage is heuristic. The factor shocks are not the fundamental causes of changes in the term structure; rather, they are sufficient statistics for fully capturing the underlying economic shocks that do cause the changes. Furthermore, statisticians use the term shocks for strictly independent events. Subsequent analysis will show that the interest rate factors are reasonably close to being, but are not precisely, serially uncorrelated.
8. The factor decompositions produced by a factor analysis are not unique. Factors can be recombined with each other in any manner so long as they remain uncorrelated and of unit variance. For example, one could let New Factor 1 = (Factor 1 + Factor 2)/√2; New Factor 2 = (Factor 1 – Factor 2)/√2; New Factor 3 = Factor 3. This process is called rotating the factors. Rotating the factors simultaneously rotates the factor loadings. The new factors are just as valid a summary of the underlying economic influences as are the original factors. However, some rotations may be more easily interpreted than the original factors produced by the factor analysis. It is common practice to first run a factor analysis and then search for rotations that make economic interpretations clear.
9. Bliss (1997) extends the Fama-Bliss method to longer maturities. Bliss also compares various term structure estimation techniques on the basis of the ability of the estimated term structures to price out-of-estimation-sample bills, notes, and bonds. Using this criterion, the Fama-Bliss method is superior to the alternatives tested.
10. Interestingly, this period does not coincide with the 1979–82 period, when interest rates were extraordinarily volatile.
11. This calculation is usually done by regressing stock returns on a market-proxy index. The resulting R²s are measures of explanatory power in regressions, comparable to the “percent of variation explained” in the factor analysis results. See Brealey and Myers (1991, Table 9-2).

In contrast to the weak stock return results, three factors explain almost all the variation in interest rate changes. While there is much debate about the modeling of interest rate movements, empirical studies of these models have no need for first reducing idiosyncratic variation through aggregation of individual securities into portfolios and thus are conducted using individual bonds or maturities of zero-coupon yields.

The top panel of Chart 1 is suggestive of some variation through time in the factor model: during the mid- to late 1970s Factor 1 made a smaller, and Factor 3 (which shows up as the difference between the top two lines) a larger contribution to explaining movements in interest rates than they did in the post-1982 period. However, by itself, this evidence is not conclusive. If a single three-factor model is fitted to the entire 1970–95 period, the percentages of variation explained by the first, first and second, and then all three factors are 80.6, 92.2, and 95.3 percent, respectively. The performance of the three-factor model, when estimated over the entire sample period, approximately equals the average performance when different factor models are estimated over shorter, twenty-four-month subperiods. This finding indicates that the factor model's performance is robust to constraining the factor loadings to be constant over long periods.

Another method for analyzing differences in models through time is to look at the estimated factors themselves. Chart 2 plots the values of the three factors estimated using a single model over the entire period. What stands out is the higher volatility of all three factors during the 1979–82 period, which corresponds to the period during which the Federal Reserve focused primarily on reducing the rate of growth of monetary aggregates, rather than targeting interest rates, in an effort to reduce inflation. Numerous studies of interest rate behavior, including Bliss and Smith (1997), have found an apparent structural shift in the process governing interest rates in this period. Chart 2, which shows a change in the volatility of the factors driving interest rates, is consistent with this result.

Analysis of Interest Rate Changes. The preceding analysis showed that three factors could explain

most of the variation in changes in interest rates, particularly since 1978. The next step is to determine, if possible, what these factors are and how to interpret them by looking at the factor loadings or the impact of each factor on each maturity. For instance, if Factor 1 changes, what happens to the three-month interest rate, to the one-year rate, and so forth? From these responses it is possible, in this case, to give an economic interpretation of the factors themselves.

The sample period from 1970 through 1995 is first divided into three subperiods suggested by the top panel of Chart 1 and by Chart 2 and observation of the changes in interest rate behavior in the early 1980s. The first period is from 1970 through September 1979, the second, from October 1979 through October 1982, and the third, from November 1982 through 1995. Table 1 presents the cumulative explanatory power of the factors over the entire period and in each of the three subperiods.

For each of the three subperiods the factor loadings are shown in Chart 3. Although they vary in the details, the factor loadings show a consistent pattern across the different periods.

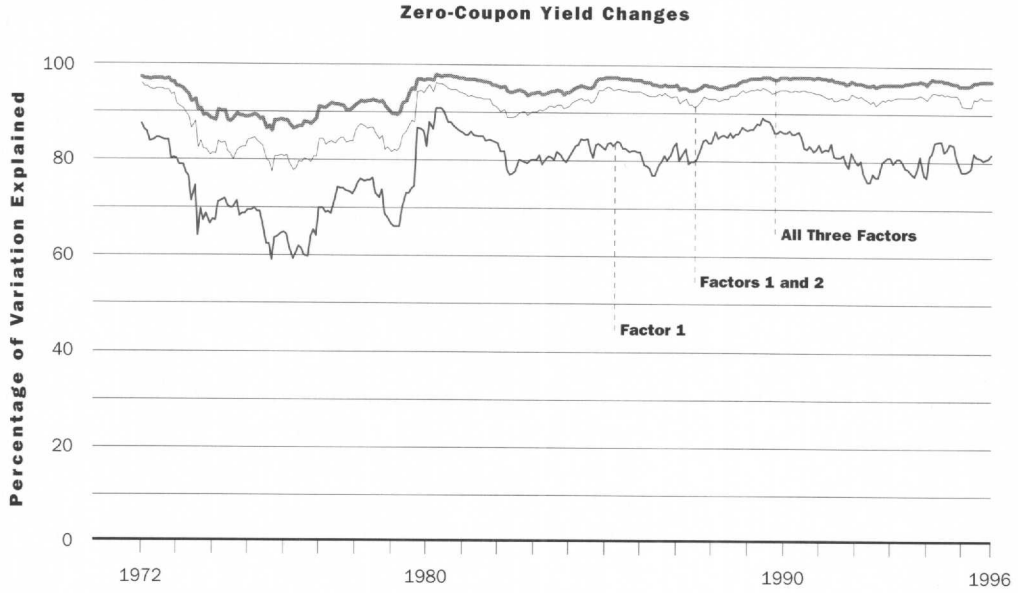
The first factor loading is very close to being constant at between 0.80 and 0.85 across all maturities. This result is true for all three subperiods. Thus, while Chart 2 shows an increase in the volatility of Factor 1 during the October 1979–October 1982 period relative to other periods, the responsiveness of interest rate changes to this factor is unchanged. The result is, of course, that interest rate changes themselves became more volatile. Since Factor 1 has roughly equal effects on all maturities, a change in Factor 1 will produce a parallel movement in all interest rates. For this reason Factor 1 can be interpreted as a "level" factor, producing changes in the overall level of interest rates. The loadings on Factor 1 are also larger in magnitude than the loadings on the other two factors. Since the variances of the factors themselves are equal by construction, this result means that more of the changes in interest rates come from the first factor than from the others.

The loadings on the second factor increase uniformly from a relatively large negative value at short maturities to

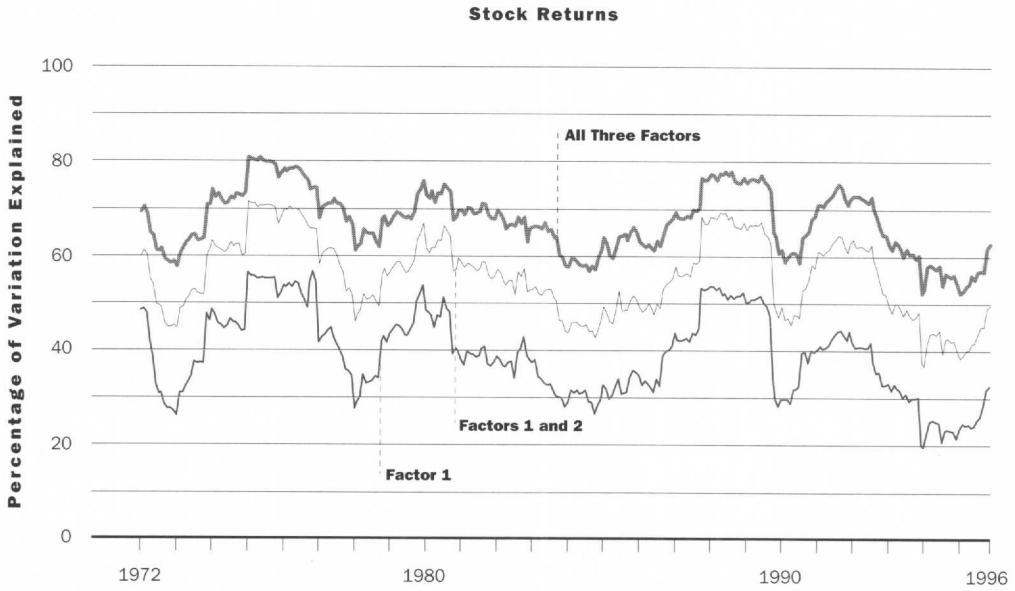
TABLE 1
Explanatory Power of the Three-Factor Model
in Various Periods

Estimated Period	Factor 1 (percent)	Factors 1 and 2 (percent)	All three factors (percent)
January 1970–December 1995	80.6	92.2	95.3
January 1970–September 1979	74.9	87.9	92.2
October 1979–October 1982	84.0	92.6	95.9
November 1982–December 1995	83.0	94.0	97.0

CHART 1 Factor Analysis



Note: Ten maturities of three and six months and one, two, three, five, seven, ten, fifteen, and twenty years; twenty-four month moving window



Note: Ten randomly selected stocks; twenty-four month moving window

CHART 2 Rotated Factor Values for Zero-Coupon Yield Changes

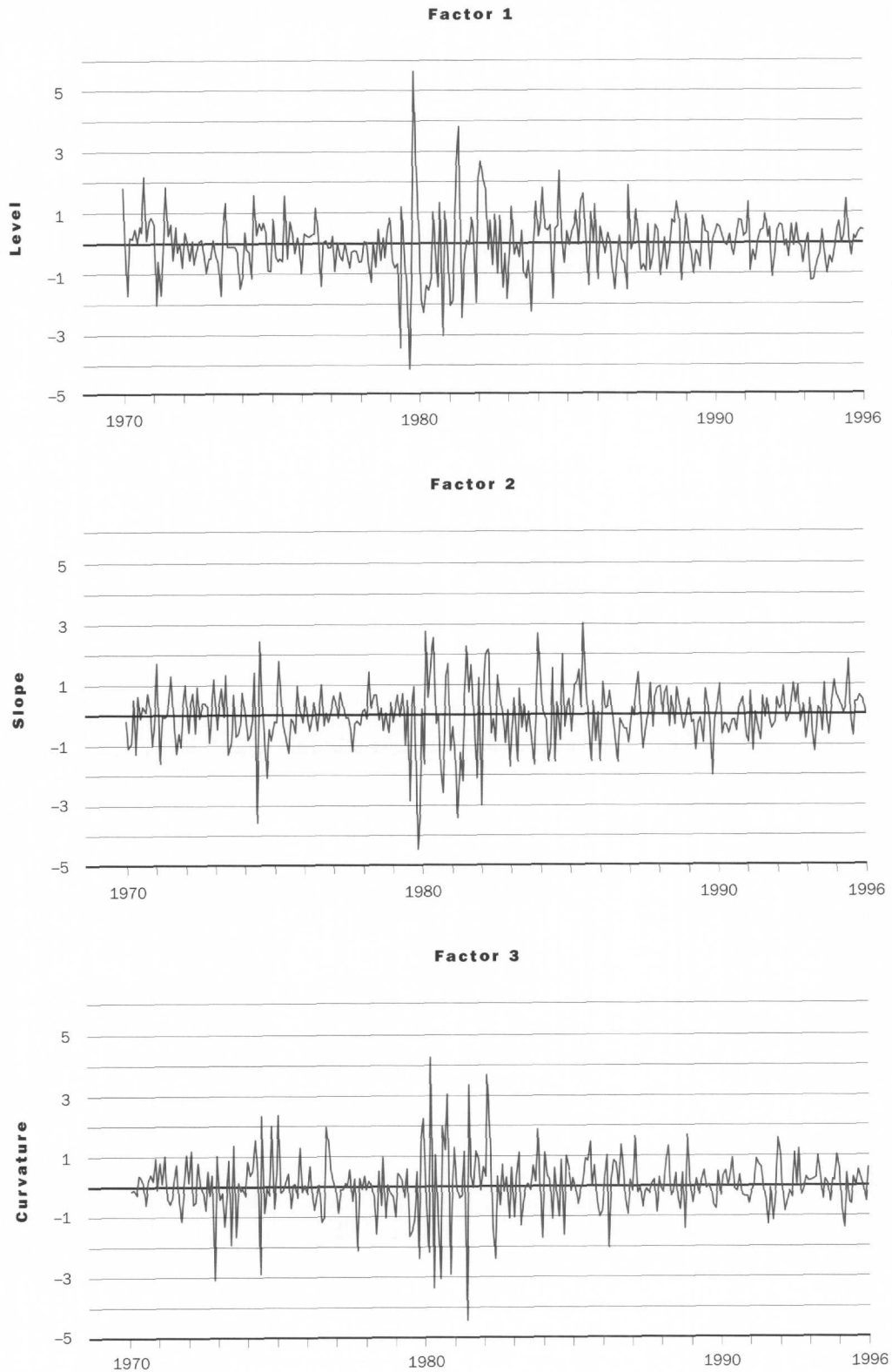


CHART 3 Rotated Factor Loadings for Zero-Coupon Yield Changes

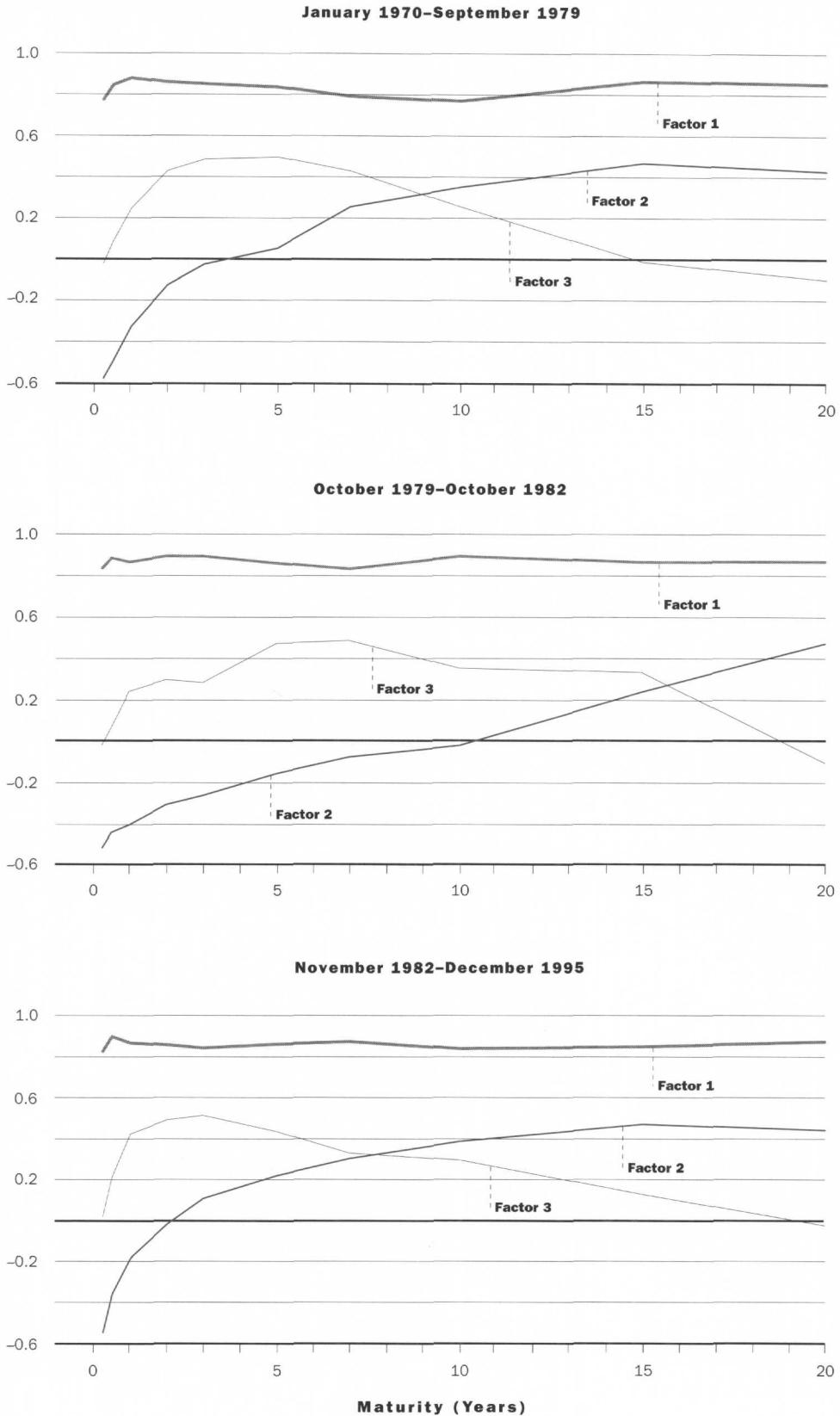
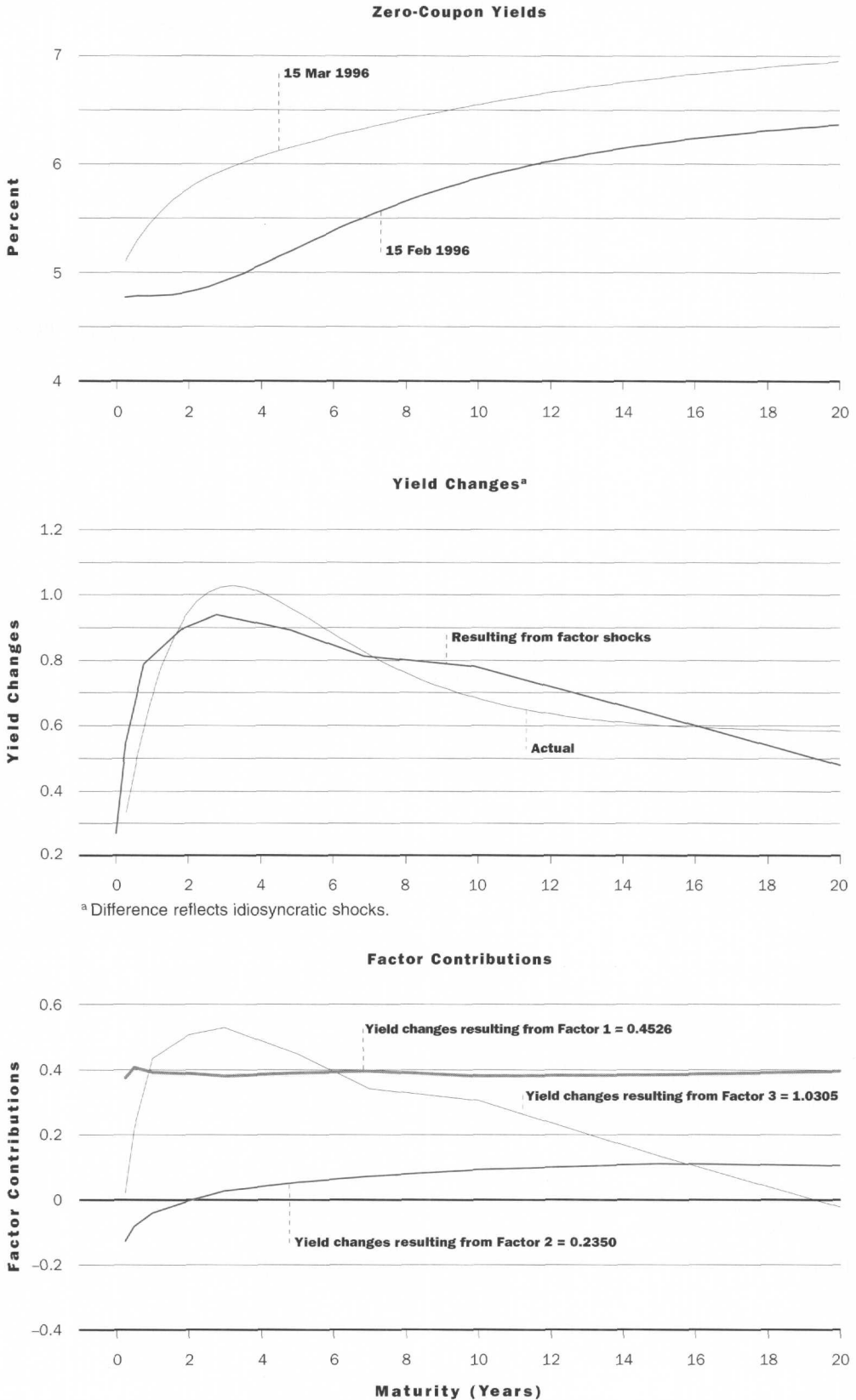


CHART 4 Decomposition of Yield Changes, February 15–March 15, 1996



a moderate positive value at the longest maturities. While the values of the short and long ends of these curves are approximately the same across subperiods, the shapes of the curve clearly vary. This pattern of increasing loadings is consistent with interpreting Factor 2 as a “slope” factor, affecting the slope of the term structure but not the average level of interest rates. Factor 2 produces movements in the long and short ends of the term structure in opposite directions (twisting the yield curve), with commensurate smaller changes at intermediate maturities.

During the period from October 1979 through October 1982, the Factor 2 loadings increase in an approximately linear fashion, centered at approximately ten years’ maturity. A change in Factor 2 then would have produced no change in the ten-year rate (since the ten-year loading on Factor 2 was then zero), shorter maturity rates would have decreased, and longer maturity rates would have risen. In both cases, the size of changes would have increased as the maturity involved was further from ten years. In contrast, in the period from November 1982 through December 1995 the Factor 2 loadings were centered at approximately two years’ maturity, and for maturities longer than ten years the effects of a change in Factor 2 are approximately constant.

Factor 3 may be interpreted as a “hump” or “curvature” factor. The loadings are zero at the shortest maturities, indicating the short rates are unaffected by Factor 3, positive for intermediate maturities, and then decline to become negative for the longest maturities. Thus a positive change in Factor 3 would tend to increase intermediate rates and decrease long rates, altering the curvature of the term structure. In the first and last subperiods the loadings peak at three to five years and decline fairly uniformly thereafter. In the intermediate period the peak is around seven years, and the loadings do not decline markedly until around fifteen years’ maturity.

A classic example of a sharp change in the shape of the term structure occurred in February and March of 1996. At the beginning of the period the market, as evidenced by federal funds futures prices, expected the Federal Reserve to continue to lower short-term rates after two consecutive 25 basis point declines in the federal funds target rate in December and January.¹² At the same time, the term structure was declining out to about two years before it sloped upward, as shown by the heavier line in the top panel of Chart 4. This “inverted hump” shape is unusual. It probably reflects concerns that the weak 0.9 percent increase in real gross

domestic product in the fourth quarter of 1995 might foreshadow a recession and that the Federal Reserve would have to cut interest rates to offset this slowdown. Early in March it became clear that budget impasses, which had shut down the government repeatedly (and particularly hampered the reporting of economic data), had been resolved, and it was becoming increasingly evident that the weak fourth quarter economic results were a temporary aberration. This positive macroeconomic news led market participants to reverse their expectations of the Federal Reserve’s near-term policy actions, and the term structure became sharply upwardly sloped out to two years before continuing to increase at a more moderate rate. The thinner line in the middle panel shows the resulting yield curve changes. Applying factor analysis to these changes, using the 1982–95 decomposition, one can compute the shocks in terms of the three factors; the estimates for Factors 1–3 are 0.4526, 0.2350, and 1.0305, respectively. Overall, there was an increase in the level of interest rates (Factor 1) of about 0.4 percent, substantially offset at the shortest maturities by an increase in the slope (Factor 2). The increase in the long-term rate is the sum of the slope change, which is positive at long maturities, and the level. The change in the curvature of the term structure from convex to concave is entirely due to the large Factor 3 shock, the curvature factor. Adding the effects of the three factors, shown individually in the bottom panel, produces the total changes in the yield curve resulting from common factors, shown as the heavier line in the middle panel. The differences between the actual yield changes and the changes due to factor shocks reflect the idiosyncratic, or maturity-specific, component in the yield changes. It can be seen that this idiosyncratic component is of second-order importance.

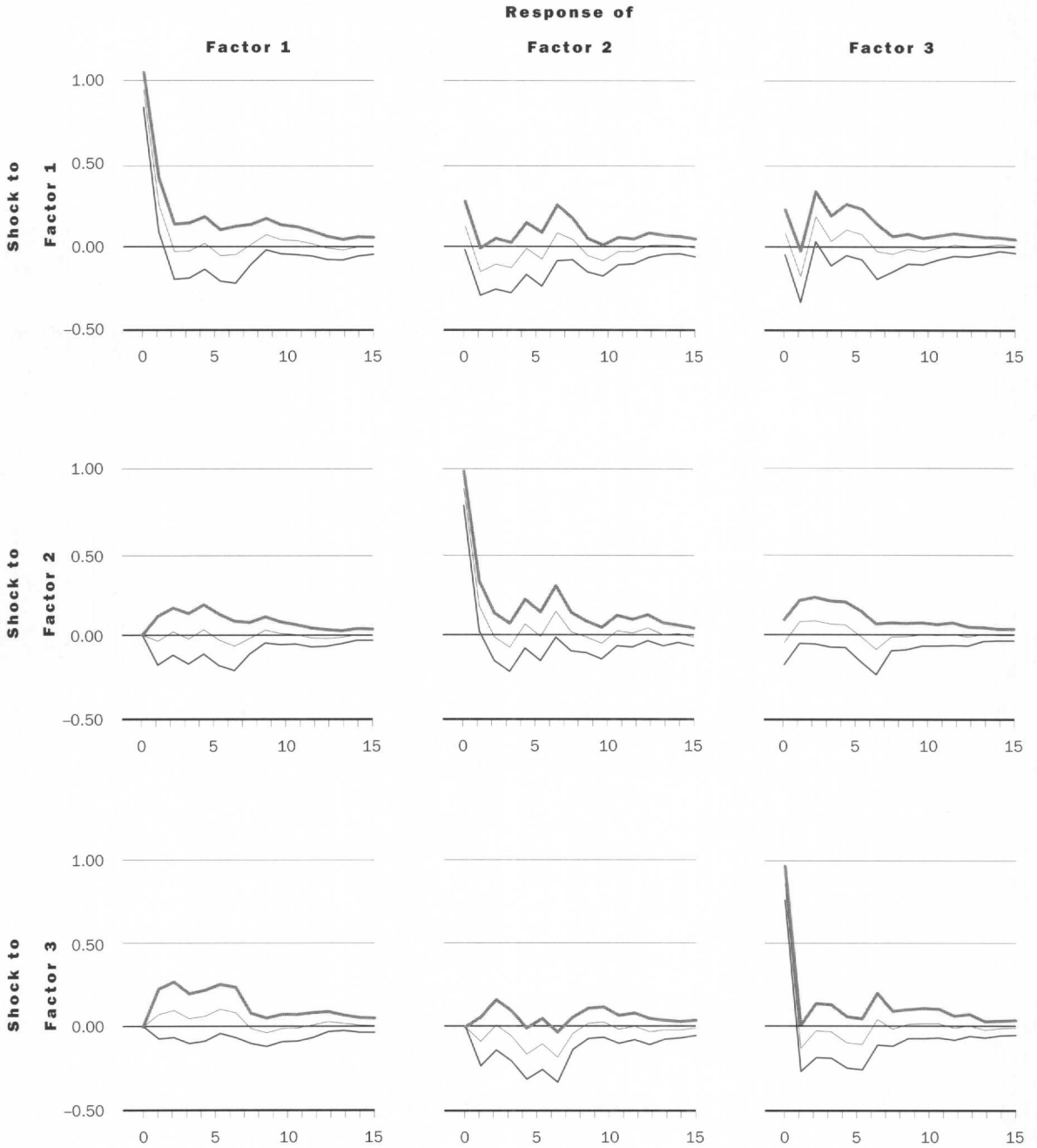
The Time-Series Behavior of Factor Changes.

By construction, the factors produced by a factor analysis are uncorrelated with each other and have identical unit variances. Nothing in the estimation procedure, however, constrains how the factors

The ability of a few factors to explain changes in interest rates is remarkably constant. Three factors can explain most of the variation, particularly since 1978.

12. Federal funds futures contracts trade on the Chicago Board of Trade. Each contract’s terminal value is tied to the average effective federal funds rate over its expiration month. Because contracts for various expiration months trade simultaneously, and because the Federal Reserve directly targets the federal funds rate in its open market operations, the market’s expectations of Federal Reserve actions can be inferred from the term structure of federal funds futures prices.

CHART 5 Impulse Responses, Six Lags



behave through time. Interest rate levels may persist (that is, positive Factor 1 shocks may tend to be followed by additional positive Factor 1 shocks). Slope increases in one period (a positive shock to Factor 2) may increase the likelihood that rates will increase in the next period (a positive shock in the next period to Factor 1). The time-series behavior of the three factors was examined using a vector autoregressive model, or VAR (see the appendix). Impulse-response functions based on the VAR are generated to show how a hypothesized single shock to one factor leads to subsequent changes in that factor and to current and subsequent changes for other factors. The estimated response functions, along with the 95 percent confidence bands, are shown in Chart 5. Thus, a positive shock of one standard deviation (+1.00) to Factor 1 at time 0 will be followed on average by a +0.25 change in Factor 1 at time 1. The error bands indicate that this follow-up response is barely significant. After time 1 the effects on subsequent outcomes of Factor 1 are negligible. The same time 0 shock to Factor 1 is associated with a slight, marginally significant, negative change in Factor 2 at time 1, after which the responses die out.¹³ Factor 3 shows a negative time 1 response and a positive time 2 response to the Factor 1 shock, both marginally significant. Thus, changes in levels of interest rates show a slight tendency to persist and to have a slight effect on changes in slope and curvature in the subsequent period. None of these effects persists beyond two periods.

Shocks to Factor 2 show the same slight tendency to persist for a single period, after which the responses die out. There is, however, no evidence of effects on Factors 1 or 3 of changes in Factor 2, either contemporaneously or subsequently. Shocks to Factor 3 do not persist even for a single period. Factor 1 shows no response to Factor 3 shocks. Factor 2 shows negative, though only marginally significant, responses at lags of four and six months.

Implications of the Factor Model

The factor analysis approach used in this paper is primarily exploratory in nature. It reveals that three factors account for a large portion of the underlying interest rate changes and hence for bond price movements. The analysis also provides a clue to the economic interpretation of those factors, suggesting how they might be related to broader economic factors.¹⁴

The factor analysis provides an ad hoc, though very effective, approach to hedging. However, the factor analysis does not constitute a coherent theoretical model of

interest rate movements. It provides only a set of stylized facts that any such model should be able to capture.

The first, and most obvious, implication of this analysis is that single-factor-based interest rate models, such as the single-factor CIR or Vasicek models, are incomplete, notwithstanding tests of these models that have failed to reject them. Single-factor models may be "good enough" for some applications such as managing portfolios of similar-maturity bonds, but they will result in hedging error when applied to complex securities, such as spread derivatives, for example. The box illustrates the relative performance of Macaulay duration and factor durations hedging for several hypothetical portfolios.

The factor analysis also explains why a simple procedure such as Macaulay duration immunization works as well as it does, despite being based on the false

premise that interest rates for all maturities always change by the same amount. While interest rates do not always move in parallel, the largest single factor in interest rate movements is a parallel shift, accounting for about 80 percent of the variation. This result helps explain why other single-factor-based hedging approaches have not done much better than Macaulay duration. Any single-factor-based model would result in changes in the term structure at all maturities being perfectly correlated, which is not in fact the case. The factor analysis shows that at least a portion of the non-parallel-shift component in changes in interest rates is uncorrelated with the parallel-shift component. Thus, it is difficult for any single-factor-based hedging technique, even one that attempts to capture nonparallel shifts, to improve on Macaulay duration.

A multifactor-based hedging strategy incorporating the results developed in this article would begin by constructing an interest rate model using the factor decomposition developed above and the evidence of the behavior of the factors themselves taken from the VAR analysis—a simple three-factor model with lags of no more than two periods would be sufficient. As the lagged-variable coefficients are only marginally significant and thus likely to be spurious, consideration should also be

The factor analysis provides an ad hoc, though very effective, approach to hedging. However, the factor analysis does not constitute a coherent theoretical model of interest rate movements.

13. Apparent significant contemporaneous correlation between Factors 1 and 2 is an artifact of the estimation method.

14. Litterman, Scheinkman, and Weiss (1991) show how term structure curvature may be related to volatility in a simple single-factor interest rate model (there factor has a different meaning from the linear factors in a factor analysis decomposition).

A Comparison of Macaulay Duration Hedging and Factor Durations Hedging

The month from February 15 to March 15, 1996, provides a vivid example of the comparative advantage of factor durations hedging over Macaulay duration hedging. Three portfolios of bonds were constructed on February 15 as follows:

- (1) Portfolio 1: A single twenty-year 8 percent coupon bond (paying coupons semiannually, as is usual)
- (2) Portfolio 2: Equal numbers of one-year and twenty-year 8 percent coupon bonds
- (3) Portfolio 3: Long positions in one unit each of one-year and twenty-year zero-coupon bonds, together with a short position in one unit of a ten-year zero-coupon bond

The face values of the bonds in the portfolios were adjusted so that the initial price of each portfolio was \$100. The portfolios were priced using the February 15 term structure and their Macaulay and factor durations, the latter computed using the November 1982–December 1995 factor loadings. See Table A.

Portfolio 1 loads heavily on the level factor. This result suggests that Macaulay duration will provide a reasonable basis for hedging. However, since the slope- and level-factor durations are not zero, hedges based on all three factors should do somewhat better. Portfolio 2 is a portfolio of coupon bonds of widely divergent maturities. Such a portfolio is likely to be sensitive to both changes in levels and changes in the slope of the term structure. Macaulay duration hedging is expected to do even less well in this instance. Portfolio 3 is a portfolio of mixed long and short positions. Its level-factor duration is approximately equal to its Macaulay duration. However, as shown by the curvature-factor duration, Portfolio 3 is particularly sensitive to changes in the curvature of the term structure. It is unlikely that Macaulay duration can capture the interest rate sensitivity of such a portfolio.

Two hedge portfolios were constructed for each portfolio. The first "Macaulay duration-matched" portfolio consisted of two zero-coupon bonds of adjacent (six months apart) maturity, chosen to match both the price and the Macaulay duration of the portfolio being hedged.¹ The second "factor durations-matched" portfolio consisted of zero-

TABLE A
Macaulay Duration versus Factor Durations

Macaulay Duration	Factor Durations		
	Level	Slope	Curvature
	Portfolio 1		
10.98	9.47	4.45	1.29
	Portfolio 2		
6.40	5.53	2.33	0.89
	Portfolio 3		
1.58	1.65	0.27	-2.07

coupon bonds of one, five, ten, and twenty years' maturity, in amounts chosen to match the price and all three factor durations of the portfolio being hedged.

Each portfolio and the associated two hedge portfolios were then repriced on March 15, 1996. An ideal hedge portfolio would have the same return over the period from February 15 to March 15 as the portfolio it is hedging. See Table B.

These results clearly show that hedges based on the three factor durations outperform hedging based on Macaulay duration. Even for the "plain vanilla" long bond (Portfolio 1), the hedging error of the Macaulay duration hedge portfolio, while small (only 0.27 percent), is more than twice that of the factor durations hedge portfolio. The Macaulay duration hedge portfolio for Portfolio 2 shows a large hedging error, greater than 1 percent, while the factor durations hedge portfolio shows a small error of only 1/20 of 1 percent. Portfolio 3's duration is short, only 1.6 years, and thus the portfolio has less price sensitivity to interest rate swings than do Portfolios 1 and 2. Nonetheless, even though price changes were small, Macaulay duration hedge portfolio missed the mark by more than 1 percent while the hedging error produced by the factor durations hedge portfolio was insignificant.

These three sample portfolios illustrate how in times of unusual interest rate movements, when the slope and curvature of the term structure are changing significantly, Macaulay duration is unable to provide a sound basis for hedging a wide variety of cash flows. It is in these cases that factor durations hedging becomes more valuable. Litterman and Scheinkman (1991) examined the ability of

1. This "near bullet" bond approximates a zero coupon of the desired duration while avoiding the need to interpolate between coupon payment dates.

TABLE B
Macaulay Duration versus Factor Durations

Actual Portfolio Return (Percent)	Macaulay Duration Hedge Portfolio		Factor Durations Hedge Portfolio	
	Return (Percent)	Hedging Error (Percent)	Return (Percent)	Hedging Error (Percent)
	Portfolio 1			
-7.41	-7.70	0.29	-7.31	-0.10
	Portfolio 2			
-4.08	-5.12	1.03	-4.03	-0.06
	Portfolio 3			
0.27	-0.79	1.07	0.27	0.0

factor durations hedging and Macaulay duration hedging to hedge weekly returns for thirteen Treasury bonds of maturities ranging from six months to nearly thirty years over the period from February 1984 through August 1988. Their

results confirm that, over a wide range of term structure movements and portfolios, factor durations hedging is significantly better than simple Macaulay duration hedging.

given to a simple three-factor model with no lags. The hedge would then construct a portfolio of short and long exposures to the three factors so that the net exposure of the portfolio to each factor is minimized.¹⁵

Conclusion

The three-factor decomposition of movements in interest rates, first uncovered by Litterman and Scheinkman (1991), is robust. The ability of the three factors to explain observed changes in interest rates is high in all subperiods studied and particularly since 1978, when they explained virtually all interest rate movements. Furthermore, the nature of the movements—level, slope, and curvature—has not changed, although the cross-sectional loadings have varied slightly. The factors themselves appear to be well behaved. There is only slight evidence of time-series or cross-factor interactions that would complicate modeling. There is

evidence of increased volatility of the factors during the Federal Reserve experiment period of October 1979 through October 1982, which is to be expected given the rise in interest rate volatility in that period.

The success of a parsimonious factor model involving interest rates contrasts with the moderate, at best, success of the same approach to modeling stock returns. This dichotomy underlies the completely different approaches to risk management used for equities and interest rate-sensitive securities. In the former case, the emphasis is on idiosyncratic risk reduction through portfolio diversification, perhaps with futures used to hedge the small systematic component of stock returns. For interest rate-sensitive securities the emphasis is on precisely balancing a portfolio to achieve the desired exposure to systematic risk factors. There is little use for portfolio diversification in managing interest rate-sensitive portfolios,

15. Two key principles govern how to hedge. The first is that, in general, risks cannot be hedged piecemeal. If stock prices and bond prices tend to move together, then it is incorrect to hedge the market risk in the stock portfolio independently of the interest rate risks in the bond portfolio. Hedging correlated risks separately results in costly redundancy in the hedges since the correlations tend to reduce the total risk of the combined portfolio below that obtained by summing the risks of the components. The exception to this rule is when the risks are uncorrelated, as are the risks associated with the factors obtained by factor analysis. In this article, the factors are the sources of interest rate risk and are not correlated with each other. Therefore, they may be hedged individually, one factor at a time, though for each factor hedging should be done across all bonds in the portfolio. This approach may not work if the portfolio contains other types of securities, such as stocks, subject to additional sources of risk.



and when it is used it is primarily to address the separate issue of credit risk where it exists.

The factor analysis shows that parsimonious interest rate models are adequate for hedging. The analysis also indicates that interest rate models should not be so parsimonious as to have only a single underlying source of uncertainty. Applying this information to hedging is fairly

straightforward. As shown in the appendix, factor durations are analogous to Macaulay duration and can be easily computed. Furthermore, the factor durations of a portfolio are the weighted averages of the portfolio components' factor durations. Thus, factor immunization is straightforward. Translating the factor structure into a rigorous interest rate model is apt to be more problematic.

APPENDIX

Mathematical Details

Factor Analysis

At each period, $t = 1, \dots, T$, we observe a p vector of variables, \mathbf{X}_t . Factor analysis assumes that these observations are linearly related to m , $m < p$, underlying unobserved factors by the following relation:

$$\begin{matrix} \mathbf{X}_t - \boldsymbol{\mu} & = & \mathbf{L} & \mathbf{F}_t & + & \boldsymbol{\varepsilon}_t \\ p \times 1 & & p \times m & m \times 1 & & p \times 1 \end{matrix} \quad (A1)$$

where

$$E(\mathbf{X}_t) = \boldsymbol{\mu}$$

$$E(\mathbf{F}_t) = \mathbf{0}$$

$$E(\boldsymbol{\varepsilon}_t) = \mathbf{0}$$

$$\text{cov}(\mathbf{F}_t) = E(\mathbf{F}_t \mathbf{F}_t') = \mathbf{I}$$

$m \times m$

$$\begin{aligned} \text{cov}(\boldsymbol{\varepsilon}_t) = E(\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t') &= \boldsymbol{\Psi} \\ & \quad p \times p \\ &= \begin{bmatrix} \Psi_1 & 0 & \dots & 0 \\ 0 & \Psi_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \Psi_p \end{bmatrix} \end{aligned}$$

The \mathbf{F}_t s are called *factors* and the \mathbf{L} s are called *factor loadings*.

In this article $p = 10$, corresponding to the maturities of the yield changes being analyzed, and t indexes the months for which observations are made. Based on the evidence in previous studies, in this article $m = 3$. Johnson and Wichern (1982, chap. 9) discuss techniques for determining the appropriate numbers of factors where these are not known ex ante.

Everything on the right-hand side of the first equation is unknown; however, the structure implies that

$$\boldsymbol{\Sigma} \equiv E[(\mathbf{X}_t - \boldsymbol{\mu})(\mathbf{X}_t - \boldsymbol{\mu})'] = \mathbf{L}\mathbf{L}' + \boldsymbol{\Psi}$$

Note that the factors, \mathbf{F}_t , do not appear, nor do the individual residual errors, $\boldsymbol{\varepsilon}_t$. Because these variables do not appear in structure of $\boldsymbol{\Sigma}$, the number of unknown parameters is reduced so that it then becomes possible to compute the factor loadings, \mathbf{L} , and idiosyncratic variances, $\boldsymbol{\Psi}$.

Two methods are widely used for estimating the elements of \mathbf{L} and $\boldsymbol{\Psi}$. The first uses the first m principal components of the estimated variance-covariance matrix, $\boldsymbol{\Sigma}$, to construct \mathbf{L} . Then $\boldsymbol{\Psi}$ is constructed from $\boldsymbol{\Sigma} - \mathbf{L}\mathbf{L}'$ by setting the off-diagonal elements to zero.¹ The second method is to assume that the residuals are multivariate-normal and to estimate the model parameters using maximum likelihood. Both methods are explained in detail in Johnson and Wichern (1982) and other standard texts on factor analysis. In this article, principal components estimation is used.

The solution, \mathbf{L} , is not unique. If \mathbf{T} is any orthogonal matrix (that is, $\mathbf{T}\mathbf{T}' = \mathbf{T}'\mathbf{T} = \mathbf{I}$), then $\mathbf{L}^* = \mathbf{L}\mathbf{T}$ is also a solution:

$$\boldsymbol{\Sigma} = \mathbf{L}^* \mathbf{L}^{*'} + \boldsymbol{\Psi} = \mathbf{L}\mathbf{T}\mathbf{T}'\mathbf{L}' + \boldsymbol{\Psi} = \mathbf{L}\mathbf{L}' + \boldsymbol{\Psi} = \boldsymbol{\Sigma}$$

Rotation also affects the factors, \mathbf{F}_t . When \mathbf{L} in equation (A1) is replaced with $\mathbf{L}^* = \mathbf{L}\mathbf{T}$, the factors become $\mathbf{F}_t^* = \mathbf{T}'\mathbf{F}_t$ so that

$$\mathbf{X}_t - \boldsymbol{\mu} = \mathbf{L}^* \mathbf{F}_t^* + \boldsymbol{\varepsilon}_t = \mathbf{L}\mathbf{T}\mathbf{T}'\mathbf{F}_t + \boldsymbol{\varepsilon}_t = \mathbf{L}\mathbf{F}_t + \boldsymbol{\varepsilon}_t$$

Rotation of the factor loadings has no effect on $\boldsymbol{\Psi}$. This indeterminacy permits rotating the original solution until factor loadings that have meaningful economic interpretations are obtained. For instance, for this article the load-

ings were first rotated so that yield changes at all maturities had approximately the same loading on the first factor. Since any perturbation to the first factor then affects all maturities equally, this factor is interpreted as a “level” shift. The interpretation of the remaining factors was obvious without further rotations.

To construct the rotation matrix, T , the following algorithm was used. Since T is 3×3 , it has nine elements. However, the requirement that it be orthogonal reduces the effective number of free parameters to three. T was built up from three orthogonal rotation matrices, each leaving one column of L (and F_t) unchanged, and recombining the two remaining columns while preserving the orthogonal structure of the resulting L^* .

$$T_1 = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 \\ -\sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_2 = \begin{bmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix}$$

$$T_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_3 & \sin\theta_3 \\ 0 & -\sin\theta_3 & \cos\theta_3 \end{bmatrix}$$

Then $T = T_1 T_2 T_3$ is an orthogonal matrix with three free parameters θ_1 , θ_2 , and θ_3 . A nonlinear optimizer was used to search over feasible values of θ_1 , θ_2 , and θ_3 (in the range $-\pi$ to $+\pi$) to find the value that minimized the variance of the first column L^* .

Once the appropriate estimated factor loadings have been obtained, the estimated factors themselves can be computed by

$$F_t = (L' \Psi^{-1} L)^{-1} L' \Psi^{-1} (X_t - \mu).$$

The properties of the estimated factors themselves can then be studied. For example, the time-series properties of the factors can be investigated, as is done in this article.

Vector Autoregression

The three factors estimated through factor analysis are, by construction, orthogonal and of unit variance (that is, $F_t' F_t = I$). However, there is no structure imposed on the time-series behavior of the factors. To investigate the possible

time-series structure of the factors, a vector-autoregressive (or VAR) model is used. The model relates current values of each of the factors to past values of all the factors.

$$F_t = A_1 F_{t-1} + \dots + A_j F_{t-j} + \eta_t$$

$m \times 1 \quad m \times m \quad m \times 1 \quad m \times m \quad m \times 1 \quad m \times 1$

where $E(\eta_t) = 0$ and $E(\eta_t' \eta_t) = I$. Normally it is advisable that this assumed restriction on the residuals be tested. However, in this case the restriction holds by construction since the factors are orthogonal. This model can be estimated using maximum likelihood methods as outlined in Hamilton (1994).

The VAR analysis shows whether there are any linear time-series relations among the factors—that is, whether a shock to one factor in this period may have an influence on another factor the next period. These relations are evidenced by nonzero elements in A_j . Equivalently, the effects through time of individual shocks on each of the factors can be examined using impulse response functions. An impulse response function uses the estimated VAR model to estimate the impact of a single, hypothetical, one-time, one-standard-deviation shock to each factor on the other factors in subsequent periods. The confidence intervals for these response functions are computed using the techniques developed in Sims and Zha (1995).

Hedging Portfolios against Factor Risks

Before beginning the discussion of hedging interest rate movements using the factor model, it will be useful to consider the simpler Macaulay duration hedging. Macaulay duration is based on an assumed single interest rate (that is, flat term structure). The price of any stream of cash flows, CF_m , $m = 1, \dots, M$, is thus²

$$P = \sum_{m=1}^M CF_m e^{-my}.$$

For a normal coupon bond, the cash flows would correspond to the individual, usually equal, coupon payments each period prior to maturity and the final coupon plus principal repayment at maturity. For a portfolio of interest rate-sensitive securities, the cash flows each period would be aggregated across securities in the portfolio. These cash flows may be either positive (for assets or long positions) or negative (for liabilities or short positions).

1. An examination of the residual matrix, $\Sigma - LL'$, to see if the off-diagonal elements are “small” before setting them to zero, is advised. If some off-diagonal elements are large, there may be additional omitted factors, and a larger value for m may be warranted. In this article, this problem did not occur.



The continuously compounded Macaulay duration is found by taking the derivative of price, P , with respect to yield, y :

$$\frac{dP}{dy} = \sum_{m=1}^M (-m) CF_m e^{-my},$$

then dividing both sides by P and expressing changes in P in terms of changes in y :

$$\frac{dP}{P} = - \left[\sum_{m=1}^M \frac{-m CF_m e^{-my}}{P} \right] dy = - \left[\sum_{m=1}^M m w_m \right] dy = -D dy.$$

Since $CF_m e^{-my}$ is the present value of the cash flow at time m , the ratio $w_m \equiv CF_m e^{-my}/P$ is the fraction of the bond's or portfolio's value due to the cash flow at time m . Thus, Macaulay duration, D , is the weighted average of the time to each cash flow, m , where the weights, w_m , are the fractions each cash flow currently contributes to the value of the total portfolio.

When securities are combined into portfolios, the duration of the portfolio is the weighted average of the durations of the assets (either individual securities or portfolios) in the portfolio, where the weights are the fractions invested in each asset. These weights may be positive or negative but must sum to unity.

Suppose two portfolios have durations D_1 and D_2 and the objective is to construct a portfolio that is immunized to changes in yields, that is, $D_p = 0$. The portfolio weights, or fractions invested in each of portfolios 1 and 2, are $w_1 \equiv D_2/(D_2 - D_1)$ and $w_2 \equiv (1 - w_1) = -D_1/(D_2 - D_1)$. If both durations are positive (or negative), then one of the basic portfolios will need to be sold short, resulting in a negative weight.

The same ideas apply to the factor structure of yield changes, although the mathematics is slightly more complicated. The change in any given zero-coupon interest rate, y_m , for maturity m , is related to the factor shocks, F_i , $i = 1, 2, 3$ (recall that factor outcomes each period are the sources of yield changes), by the factor loading, $L_{i,m}$, appropriate to maturity m , for that zero coupon interest rate:

$$dy_m = \sum_{i=1}^3 L_{i,m} F_i.$$

The change in value of a portfolio of interest rate-sensitive cash flows, P , is a function of the changes in each interest rate indicated by

$$\begin{aligned} dP &= \sum_{m=1}^M \frac{\partial P}{\partial y_m} dy_m = \sum_{m=1}^M (-m) CF_m e^{-my_m} dy_m \\ &= - \sum_{m=1}^M \left[m CF_m e^{-my_m} \sum_{i=1}^3 L_{i,m} F_i \right]. \end{aligned}$$

Dividing through by P and rearranging yields

$$\begin{aligned} \frac{dP}{P} &= - \sum_{i=1}^3 \left[\sum_{m=1}^M m \frac{CF_m e^{-my_m}}{P} L_{i,m} \right] F_i = - \sum_{i=1}^3 \left[\sum_{m=1}^M m w_m L_{i,m} \right] F_i \\ &= - \sum_{i=1}^3 D_i F_i. \end{aligned}$$

To summarize, Macaulay duration, D , is a measure of the portfolio's or bond's sensitivity to changes in the (single) interest rate, y . Macaulay duration is computed by taking the weighted average time-to-cash flow where the weights are the present values of the cash flows divided by the total value of the portfolio. Factor durations, D_i , are analogous. They measure the sensitivity of a portfolio's value to each of the factors. Factor durations are the weighted average of the time-to-cash flows multiplied by the factor loadings appropriate to the horizon of the cash flow. As in the case of Macaulay duration, the weights are the present value of each cash flow divided by the value of the portfolio.

Factor durations combine linearly. If two portfolios have prices P_1 and P_2 and associated factor durations, D_i^1 and D_i^2 , $i = 1, 2, 3$, then the factor durations of the portfolio will be

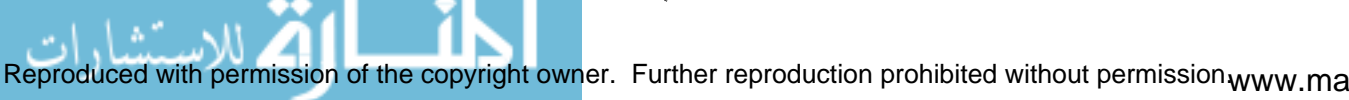
$$D_i^p = \frac{P_1}{P_1 + P_2} D_i^1 + \frac{P_2}{P_1 + P_2} D_i^2,$$

or more generally

$$D_i^p = \sum_{j=1}^N x_j D_i^j,$$

where the x_j are the fractions of total portfolio value represented by the value of each component (which may be individually positive or negative but must sum to unity). Immunization in this context requires selecting portfolio weights, x_j , so that all the combined portfolio factor sensitivities are zero. The three-factor interest rate decomposition presented here requires four bonds or bond portfolios with sufficiently different (that is, linearly independent) factor durations to serve as building blocks. Then four simultaneous equations, one for each factor and one to guarantee that the weights sum to unity, are solved to arrive at the weights for each element of the portfolio.

2. For notational simplicity the t denoting time of observation, P_t or y_t , or arrival of shocks, F_t , is omitted in the following.



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